Quiz 2.2: Sample Answers

Note: these questions ask us to find the derivative using the limit definition, so we need to solve it without using the derivative rules we learned later.

1. Find the slope of the tangent line to the function $\frac{x+3}{x+15}$ at the point $x = 4$. The slope is given by the limit formula:

$$\lim_{x\to 4} \frac{f(x) - f(4)}{x - 4} \quad = \quad \lim_{x\to 4} \frac{\frac{x+3}{x+15} - \frac{7}{19}}{x - 4}$$

   We then find bring the fractions together by finding a common denominator:

$$\begin{aligned}
&= \lim_{x\to 4} \frac{19x + 57 - 7x - 105}{(x + 15)(19)(x - 4)} \\
&= \lim_{x\to 4} \frac{12x - 48}{(x + 15)(19)(x - 4)} \\
&= \lim_{x\to 4} \frac{12(x - 4)}{(x + 15)(19)(x - 4)} \\
&= \lim_{x\to 4} \frac{12}{(x + 15)(19)} \\
&= \frac{12}{(4 + 15)(19)} \\
&= \frac{12}{361}
\end{aligned}$$

2. Find the equation of the tangent line to $f(x) = 3 + x + 3x^2$ at $x = 3$. We first need to find the slope of the tangent line at $x = 3$:

$$\begin{aligned}
\lim_{x\to 3} \frac{f(x) - f(3)}{x - 3} \quad &= \quad \lim_{x\to 3} \frac{(3 + x + x^2) - (3 + 3 + 3(3)^2}{x - 3} \\
&= \quad \lim_{x\to 3} \frac{3x^2 + x - 30}{x - 3}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{x\to 3} \frac{3(x^2 + x/3 - 10)}{x-3} \\
&= \lim_{x\to 3} \frac{3(x-3)(x+10/3)}{x-3} \\
&= \lim_{x\to 3} 3(x+10/3) \\
&= 3(3+10/3) \\
&= 19
\end{aligned}
$$

So we have slope $m = 19$, and the point $x = 3$, $y = f(x) = 3 + 3 + 3(3)^2 = 33$. We set these into the equation $y = mx + b$ to solve for b:

$$33 = 19(3) + b \Rightarrow 33 = 57 + b \Rightarrow b = -24$$

So the equation of the tangent line at $x = 3$ is $y = 19x - 24$.

3. Find the equation of the tangent line to $f(x) = 3\sqrt{x+2}$ at $x = 4$. We first need to find the slope of the tangent line at $x = 4$. For square root functions, it is generally easier to use the $h \to 0$ limit definition.

$$
\begin{aligned}
\lim_{h\to 0} \frac{f(4+h) - f(4)}{h} &= \lim_{h\to 0} \frac{3\sqrt{4+h+2} - 3\sqrt{4+2}}{h} \\
&= \lim_{h\to 0} \frac{3\sqrt{h+6} - 3\sqrt{6}}{h}
\end{aligned}
$$

We then multiply and divide by the conjugate, $3\sqrt{h+6} + 3\sqrt{6}$:

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{3\sqrt{h+6} - 3\sqrt{6}}{h} \left(\frac{3\sqrt{h+6} + 3\sqrt{6}}{3\sqrt{h+6} + 3\sqrt{6}} \right) \\
&= \lim_{h\to 0} \frac{9(h+6) - 9(6)}{h(3\sqrt{h+6} + 3\sqrt{6})} \\
&= \lim_{h\to 0} \frac{9h}{3h(\sqrt{h+6} + \sqrt{6})} \\
&= \lim_{h\to 0} \frac{3}{\sqrt{h+6} + \sqrt{6}} \\
&= \frac{3}{\sqrt{0+6} + \sqrt{6}} \\
&= \frac{3}{2\sqrt{6}}
\end{aligned}
$$

So we have slope $m = \frac{3}{2\sqrt{6}}$, and the point $x = 4$, $y = f(x) = 3\sqrt{4+2} = 3\sqrt{6}$. We set these into the equation $y = mx + b$ to solve for b:

$$3\sqrt{6} = \frac{3}{2\sqrt{6}}(4) + b \Rightarrow 3\sqrt{6} = \frac{6}{\sqrt{6}} + b \Rightarrow 3\sqrt{6} = \sqrt{6} + b \Rightarrow b = 2\sqrt{6}$$

So the equation of the tangent line at $x = 4$ is $y = \frac{3}{2\sqrt{6}}x + 2\sqrt{6}$.

4. Find the velocity of an object at $t = 3$, displacement is $s = t^3/7 - 5t^2/21$. Again, we use the limit equation:

$$\begin{aligned}
\lim_{x\to 3} \frac{f(x) - f(3)}{x-3} &= \lim_{x\to 3} \frac{(x^3/7 - 5x^2/21) - ((3)^3/7 - 5(3)^2/21)}{x-3} \\
&= \frac{1}{21} \lim_{x\to 3} \frac{(3x^3 - 5x^2) - 36}{x-3} \\
&= \frac{1}{21} \lim_{x\to 3} \frac{3x^3 - 5x^2 - 36}{x-3}
\end{aligned}$$

We now need to factor $3x^3 - 5x^2 - 36$. We can easily check that $x = 3$ is a root of this polynomial, so we know we can divide it by $(x-3)$. We then use long division to find $3x^3 - 5x^2 - 36 = (x-3)(3x^2 + 4x + 12)$. So the above becomes:

$$\begin{aligned}
&= \frac{1}{21} \lim_{x\to 3} \frac{(x-3)(3x^2 + 4x + 12)}{x-3} \\
&= \frac{1}{21} \lim_{x\to 3} 3x^2 + 4x + 12 \\
&= \frac{1}{21}[3(3)^2 + 4(3) + 12] \\
&= \frac{17}{7}
\end{aligned}$$

So the velocity is $17/7$.